Métodos Matemáticos de Bioingeniería

Grado en Ingeniería Biomédica

Lecture 17

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Outline

Outline

General Notion of an Integral of a Function of Two Variables

- We define a more general notion of an integral of a function of two variables that will allow us to describe

  1. Integrals of arbitrary functions

     Functions that are not necessarily nonnegative or continuous

  2. Integrals over arbitrary regions in the plane

     Rather than integrals over rectangles only

- In case 1. we will see that there is a key connection between the notion of an iterated integral and a double integral

  Fubini's Theorem

Definition 2.1: Partition of a Rectangle

- Consider the (closed) rectangle

$$R = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq d\}$$

- We also denote R as

$$R = [a, b] \times [c, d]$$

- A partition of R of order n consists of two collections of partition points
- This two collections break up R into a union of n^2 subrectangles
- More specifically, for $i, j = 0, \ldots, n$

$$\begin{aligned} a &= x_0 < x_1 < \cdots < x_{i-1} < x_i < \cdots < x_n = b \\ c &= y_0 < y_1 < \cdots < y_{j-1} < y_j < \cdots < y_n = d \end{aligned}$$

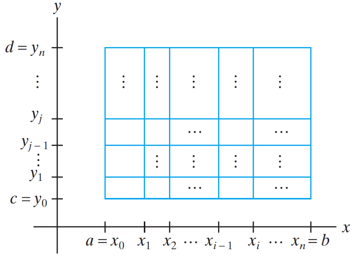
Definition 2.1: Partition of a Rectangle

$$
\begin{aligned}
R &= [a,b]\times[c,d]; \quad \text{For } i,j = 0,\ldots,n \\
a &= x_0 < x_1 < \cdots < x_{i-1} < x_i < \cdots < x_n = b \\
c &= y_0 < y_1 < \cdots < y_{j-1} < y_j < \cdots < y_n = d
\end{aligned}
$$

- For $i, j = 0, \ldots, n$ we denote

$$\Delta x_i = x_i - x_{i-1} \quad \text{and} \quad \Delta y_j = y_j - y_{j-1}$$

The width and height (respectively)
of the *ij*th subrectangle



Definition 2.2: Riemann sum

- Suppose that f is any function defined on $R = [a, b] \times [c, d]$
- Assume we partition R in some way
- Let $\mathbf{c}_{ij}$ be any point in the subrectangle

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j], \quad i, j = 1, \ldots, n$$

- Consider the quantity

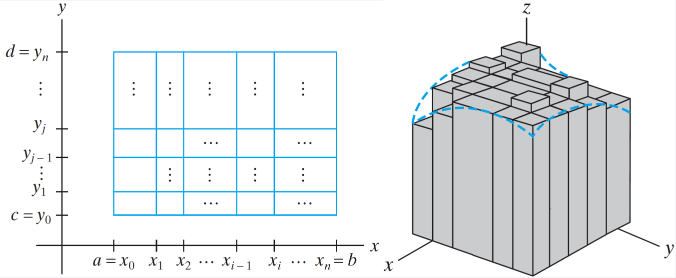
$$S = \sum_{i,j=1}^{n} f(\mathbf{c}_{ij}) \Delta A_{ij}$$

- This quantity is called a Riemann sum of f on R corresponding to the partition
- $\Delta A_{ij} = \Delta x_i \Delta y_j$ is the area of R_{ij}

Definition 2.2: Riemann sum

$$
\begin{aligned}
R &= [a,b]\times[c,d], \quad R_{ij} = [x_{i-1},x_i]\times[y_{j-1},y_j], \quad i,j=1,\dots,n \\
S &= \sum_{i,j=1}^{n} f(\mathbf{c}_{ij})\Delta A_{ij}, \quad \Delta A_{ij} = \Delta x_i \Delta y_j
\end{aligned}
$$

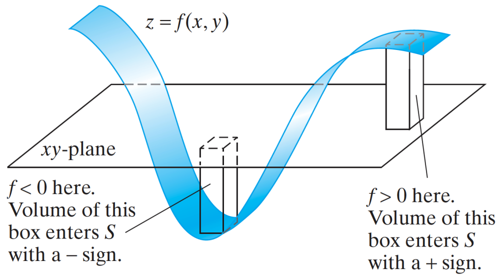
- Case 1: Suppose f happens to be nonnegative on R
- Then, S can be considered to be an approximation to the volume under the graph of f over R



Definition 2.2: Riemann sum

$$R = [a,b]\times[c,d], \quad R_{ij} = [x_{i-1},x_i]\times[y_{j-1},y_j], \quad i,j = 1,\ldots,n$$
$$S = \sum_{i,j}^{n} f(\mathbf{c}_{ij})\Delta A_{ij}, \quad \Delta A_{ij} = \Delta x_i \Delta y_j$$

- Case 2: Suppose f is not necessarily nonnegative on R
- Then, the Riemann sum S is a signed sum of such volumes (some $f(\mathbf{c}_{ij}) < 0$ and other $f(\mathbf{c}_{ij}) > 0$).



Definition 2.3: Double Integral

- The double integral of f on R is the limit of the Riemann sum S as the dimensions Δx_i and Δy_j of the subrectangles R_{ij} all approach zero

$$\int\int_R f\ dA = \lim_{\text{all } \Delta x_i, \Delta y_j \to 0} \sum_{i,j=1}^{n} f(\mathbf{c}_{ij}) \Delta x_i \Delta y_j$$

- The double integral is well defined provided that this limit exists

Remarks

- When $\int\int_R f\ dA$ exists, we say that f is integrable on R
- There are different notations for the double integral

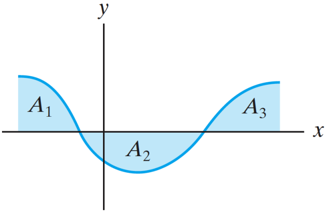
$$\int\int_R f\ dA = \int\int_R f(x,y)\ dA = \int\int_R f(x,y)\ dxdy$$

Geometric Interpretation: single-variable case

- Consider the case of a single-variable definite integral

$$\int_a^b f(x)\,dx$$

- From a geometric point of view, it can be used to compute the net area under the graph of the curve $y = f(x)$



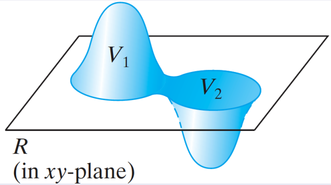
$$\int_a^b f(x)\,dx = A_1 - A_2 + A_3$$

Geometric Interpretation: two-variable case

- Consider the case of a double integral

$$\int\int_R f \; dA$$

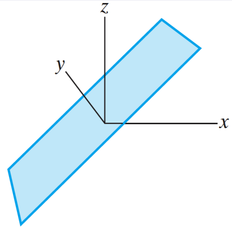
- From a geometric point of view, it can be used to compute the net volume under the graph of $z = f(x, y)$



$$\int\int_R f \; dA = V_1 - V_2$$

Example 2

- Consider the partition $R = [-2, 2] \times [-1, 3]$
- We determine the value of
$$\int\int_R x\, dA$$
- Here the integrand is
$$f(x, y) = x$$
- The portion of the plane is positioned so that exactly half of it lies above the xy-plane and half below.



Example 2

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- We determine the value of

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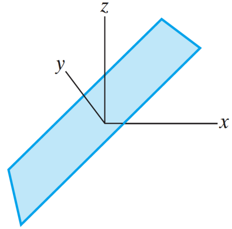


$$\int\int_R x \, dA = 0, \quad \text{(the net volume under the graph of } z = x\text{)}$$

Example 2

- Consider the partition $R = [-2, 2] \times [-1, 3]$
- We determined the value of
$$\int\int_R x \, dA = 0$$

Remark

- In general, it is difficult to use Definition 2.3 in practice to determine the integrability of a funtion
- We should be able to calculate the limit of Riemann sums over all possible partitions
- The following theorem provides an easy criterion for integrability

Theorem 2.4

- If f is continuous on the closed rectangle R, then

$$\int\int_R f \; dA$$

exists

Example 2 revisited

- In Example 2, $f(x, y) = x$ is a continuous function
- Hence, it is integrable by Theorem 2.4
- The symmetry arguments used in the example then show that

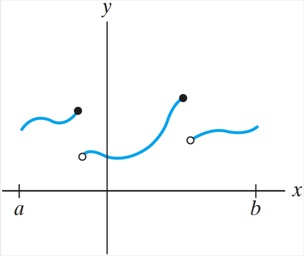
$$\int\int_R x \; dA = 0$$

Piecewise continuous functions: single-variable case

- Continuous functions are not the only examples of integrable functions
- In the case of a function of a single variable, piecewise continuous functions are also integrable
- Recall that a function $f(x)$ is piecewise continuous on the closed interval $[a, b]$ if
  - f is bounded on $[a, b]$, and
  - It has at most finitely many points of discontinuity on the interior of $[a, b]$

Piecewise continuous functions: single-variable case

- Continuous functions are not the only examples of integrable functions
- In the case of a function of a single variable, piecewise continuous functions are also integrable
- Its graph consists of finitely many continuous "chunks"



- For a function of two variables, the following result generalizes Theorem 2.4

Theorem 2.5: two-variable case

- If f is bounded on R and if the set of discontinuities of f on R has zero area, then

$$\int\int_R f \; dA$$

exists

Remarks

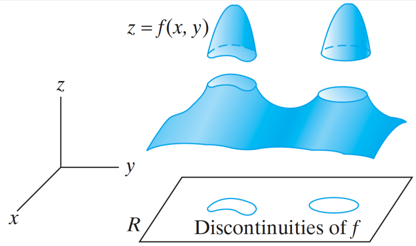
- To say that a set X has zero area means that
  - We can cover X with rectangles $R_1, R_2, \ldots, R_n, \ldots$, and
  - The sum of their areas can be made arbitrarily small

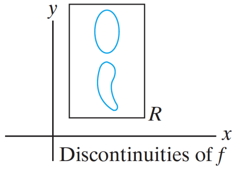
Theorem 2.5: two-variable case

- If f is bounded on R and if the set of discontinuities of f on R has zero area, then
$$\int\int_R f\, dA$$
exists

Remarks

- Zero area set:





Theorem 2.5: two-variable case

- If f is bounded on R and if the set of discontinuities of f on R has zero area, then

$$\int\int_R f\ dA$$

exists

Remarks

- Theorems 2.4 and 2.5 check that a given integral exists, but they don't provide the numerical value of the integral
- To mechanize the evaluation of double integrals, we will use Fubini's Theorem

Outline

Fubini's Theorem

- Let f be bounded on $R = [a, b] \times [c, d]$
- Assume that the set S of discontinuities of f on R has zero area.
- Then,

$$\int\int_R f \, dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$

Remarks

- Fubini's theorem demonstrates that under certain assumptions
  - The double integral over a rectangle can be calculated by using iterated integrals
  - The order of integration for the iterated integral does not matter

Example 3

- We revisit Example 2, where $R = [-2, 2] \times [-1, 3]$ and $f(x, y) = x$
- By Fubini's theorem, we calculate

$$\int\int_R x\, dA = \int_{-2}^{2}\int_{-1}^{3} x\, dy dx = \int_{-2}^{2}\left(xy\Big|_{y=-1}^{y=3}\right) dx$$
$$= \int_{-2}^{2} x(3-(-1))dx = \int_{-2}^{2} 4x\, dx = \; 2x^2\Big|_{-2}^{2} = 8 - 8 = 0$$

- It is easy to check that also

$$\int\int_R x\, dA = \int_{-1}^{3}\int_{-2}^{2} x\, dx dy = 0$$

Proposition 2.7: Properties of the Integral

- Suppose that f and g are both integrable on the closed rectangle R
  1. $f + g$ is also integrable on R and
  $$\int\int_R (f+g)\, dA = \int\int_R f\, dA + \int\int_R g\, dA$$
  2. cf is also integrable on R, where $c \in \mathbb{R}$ is any constant, and
  $$\int\int_R cf\, dA = c\int\int_R f\, dA$$
- Properties 1 and 2 are called the linearity properties of the double integral

Proposition 2.7: Properties of the Integral

- Suppose that f and g are both integrable on the closed rectangle R

  3. If $f(x, y) \leq g(x, y)$ for all $(x, y) \in R$, then

  $$\int\int_R f(x, y)\, dA \leq \int\int_R g(x, y)\, dA$$

  4. $|f|$ is also integrable on R and

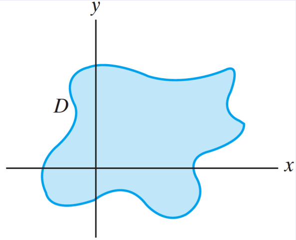
  $$\left|\int\int_R f\, dA\right| \leq \int\int_R |f|\, dA$$

- Property 3 is known as monotonicity.

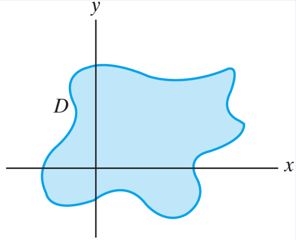
Outline

Elementary Regions

- We would like to understand how to define the integral of a function over an arbitrary bounded region D in the plane
$$\int\int_D f \; dA$$
- Ideally, we would like to give a precise definition when D is the amoeba-shaped blob shown in figure:



Elementary Regions



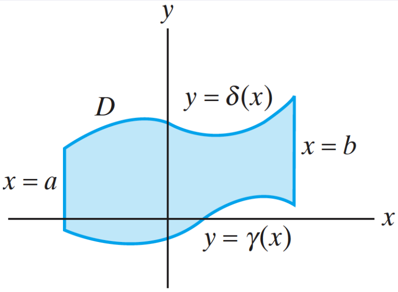
- Unfortunately, this is not possible with the techniques studied so far.
- Instead, we shall consider only certain special regions.
- And we shall assume that the integrand f is continuous over the region of integration.

Definition 2.8

- D is an elementary region in the plane if it can be described as a subset of $\mathbb{R}^2$ of one of the following three types:
  - Type 1

$$D = \{(x, y) \mid \gamma(x) \leq y \leq \delta(x), a \leq x \leq b\},$$

where γ and δ are continuous on $[a, b]$

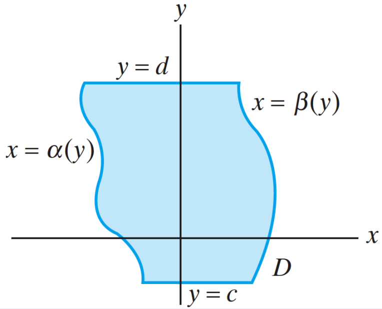


Definition 2.8

- D is an elementary region in the plane if it can be described as a subset of $\mathbb{R}^2$ of one of the following three types:
  - Type 2

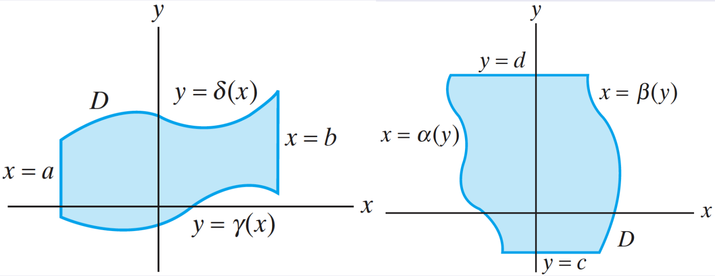
$$D = \{(x, y) \mid \alpha(y) \leq x \leq \beta(y), c \leq y \leq d\},$$

where α and β are continuous on $[c, d]$

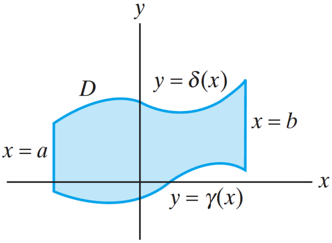


Definition 2.8

- D is an elementary region in the plane if it can be described as a subset of $\mathbb{R}^2$ of one of the following three types:
  - Type 3: the regions D that can be described as both type 1 and type 2 regions.



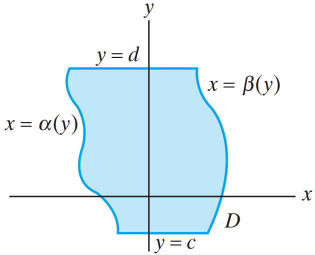
Definition 2.8



Remarks

- A type 1 elementary region D has a boundary ∂D consisting of
  - Straight segments (possibly single points) on the left and on the right, and
  - Graphs of continuous functions of x on the top and on the bottom

Definition 2.8

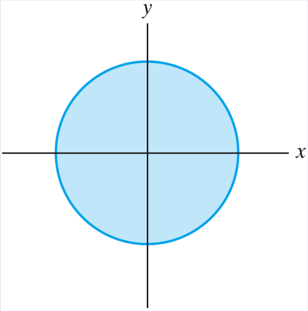


Remarks

- A type 2 elementary region D has a boundary ∂D that is
  - Straight on the top and bottom, and
  - Consists of graphs of continuous functions of y on the left and right

Example 4

- Consider the unit disk $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$

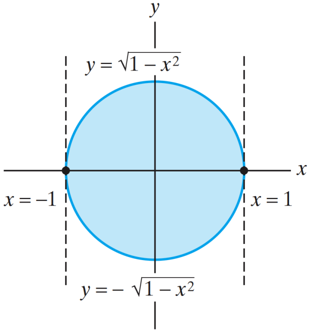


- It is an example of a type 3 elementary region

Example 4

- Consider the unit disk $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$
- It is a type 1 region since

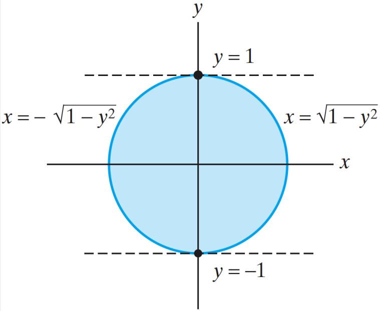
$$D = \{(x, y) \mid -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}, \, -1 \leq x \leq 1\}$$



Example 4

- Consider the unit disk $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$
- It is also a type 2 region since

$$D = \{(x, y) \mid -\sqrt{1 - y^2} \leq x \leq \sqrt{1 - y^2}, -1 \leq y \leq 1\}$$



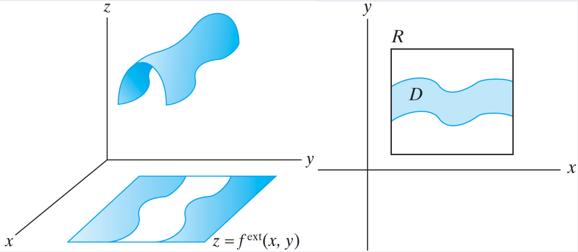
Definition 2.9

- Consider the double integral

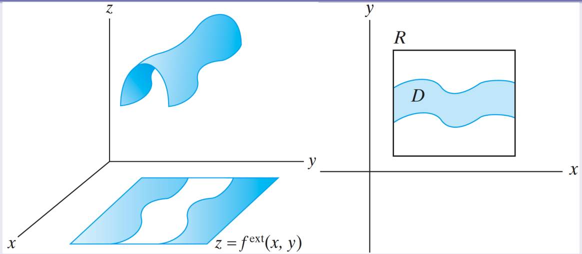
$$\int\int_D f \, dA$$

- Assume D is an elementary region and f is continuous on D
- We construct a new function f^{ext}, the extension of f, by

$$f^{ext}(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \notin D \end{cases}$$



Definition 2.9

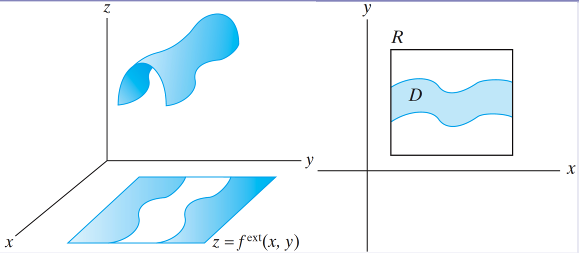


- We construct a new function f^{ext}, the extension of f , by

$$f^{ext}(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \notin D \end{cases}$$

- Note that, in general, f^{ext} will not be continuous
- But the discontinuities of f^{ext} will all be contained in ∂D, which has no area

Definition 2.9

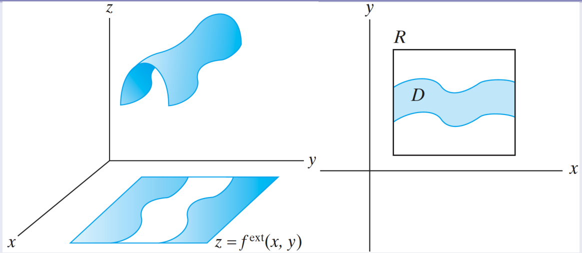


- We construct a new function f^{ext}, the extension of f , by

$$f^{ext}(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \notin D \end{cases}$$

- Hence, by Theorem 2.5, f^{ext} is integrable on any closed rectangle R that contains D

Definition 2.9



- Under the previous assumptions and notation, if R is any rectangle that contains D, we define

$$\int\int_D f\ dA = \int\int_R f^{ext}\ dA$$

Theorem 2.10

- Let D be an elementary region in $\mathbb{R}^2$ and f a continuous function on D
  1. If D is of type 1, then
  $$\int\int_D f\ dA = \int_a^b \int_{\gamma(x)}^{\delta(x)} f(x,y)dydx$$
  2. If D is of type 2, then
  $$\int\int_D f\ dA = \int_c^d \int_{\alpha(y)}^{\beta(y)} f(x,y)dxdy$$

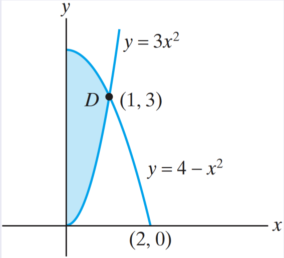
Remark

- Theorem 2.10 provides an explicit way to evaluate double integrals over elementary regions using iterated integrals

Example 5

- Let D be the region bounded by the y-axis and the parabolas

$$y = 3x^2, \quad y = 4 - x^2$$

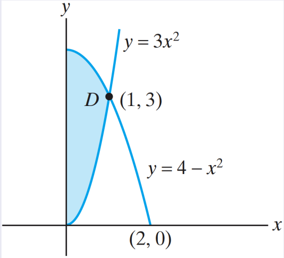


- Note that the parabolas intersect at the point $(1, 3)$
- Since D is a type 1 elementary region, we may use Theorem 2.10 with $f(x, y) = x^2 y$

Example 5

- Let D be the region bounded by the y-axis and the parabolas

$$y = 3x^2, \quad y = 4 - x^2$$



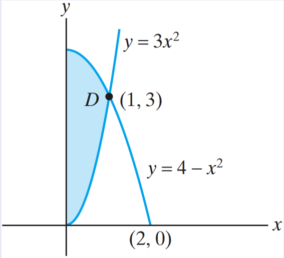
$$\int\int_D x^2 y \, dA = \int_0^1 \int_{3x^2}^{4-x^2} x^2 y \, dy\, dx$$

- The limits for the first (inside) integration come from the y-values of the top and bottom boundary curves of D

Example 5

- Let D be the region bounded by the y-axis and the parabolas

$$y = 3x^2, \quad y = 4 - x^2$$



$$\int\int_D x^2 y \, dA = \int_0^1 \int_{3x^2}^{4-x^2} x^2 y \, dydx$$

- The limits for second (outside) integration are the constant x-values corresponding to the straight left and right sides of D

Example 5

- Let D be the region bounded by the y-axis and the parabolas

$$y = 3x^2, \quad y = 4 - x^2$$

- Then,

$$\int\int_D x^2 y \, dA = \int_0^1 \int_{3x^2}^{4-x^2} x^2 y \, dydx = \int_0^1 \left(\frac{x^2 y^2}{2} \right) \Bigg|_{y=3x^2}^{y=4-x^2} dx$$

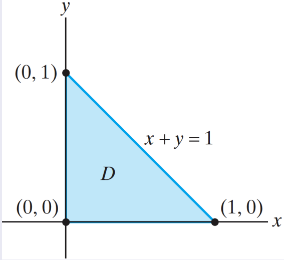
$$= \int_0^1 \frac{x^2}{2} \left(\left(4 - x^2\right)^2 - \left(3x^2\right)^2 \right) dx$$

$$= \frac{1}{2} \int_0^1 x^2 \left(16 - 8x^3 + x^4 - 9x^4\right) dx = \int_0^1 \left(8x^2 - 4x^4 - 4x^6\right) dx$$

$$= \frac{8}{3} - \frac{4}{5} - \frac{4}{7} = \frac{136}{105}$$

Example 6

- Let D be the region having a triangular border shown in figure



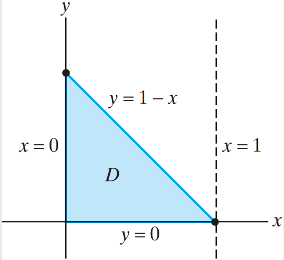
- We calculate

$$\int\int_D (1 - x - y)\, dA$$

- Note that D is a type 3 elementary region, so there should be two ways to evaluate the double integral

Example 6

- Let us consider D as a type 1 elementary region



- Then, we can apply part 1. of Theorem 2.10

$$\int\int_D (1 - x - y)\, dA = \int_0^1 \int_0^{1-x} (1 - x - y) dy dx$$

Example 6

- Then, we can apply part 1. of Theorem 2.10

$$
\begin{aligned}
\int\int_D (1-x-y)\,dA &= \int_0^1 \int_0^{1-x} (1-x-y)dydx \\
&= \int_0^1 \left(y - xy - \frac{y^2}{2} \right) \Bigg|_{y=0}^{y=1-x} dx \\
&= \int_0^1 \left((1-x) - (1-x) - \frac{(1-x)^2}{2} \right) dx \\
&= \int_0^1 \frac{(1-x)^2}{2} dx = -\frac{1}{6}(1-x)^3 \Bigg|_0^1 = \frac{1}{6}
\end{aligned}
$$

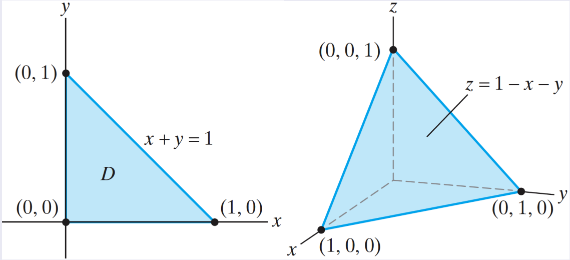
- We could obtain the same result considering D as a type 2 elementary region

Example 6

- Thus, we have obtained

$$\int\int_D (1 - x - y)\, dA = \frac{1}{6}$$

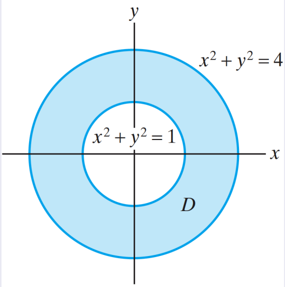
- It represents the volume under the graph of $z = 1 - x - y$ over the triangular region D



- This double integral represents the volume of a tetrahedron

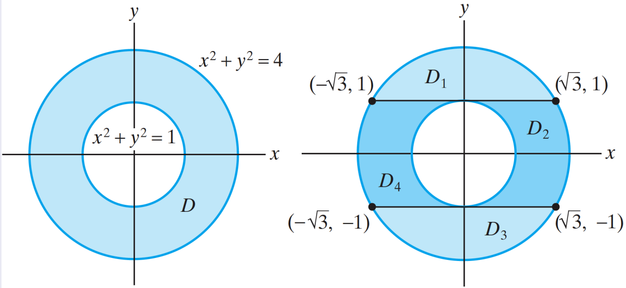
Example 7

- Let D be the annular region between the two concentric circles of radius 1 and 2 shown in figure



- Then, D is not an elementary region
- But we can break D up into four subregions that are of elementary type

Example 7

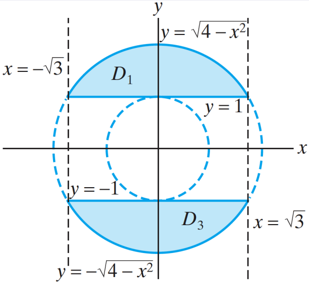


- If $f(x, y)$ is any function of two variables that is continuous (hence integrable) on D, then

$$\int\int_D f\ dA = \int\int_{D_1} f\ dA + \int\int_{D_2} f\ dA + \int\int_{D_3} f\ dA + \int\int_{D_4} f\ dA$$

Example 7

- For the type 1 subregions

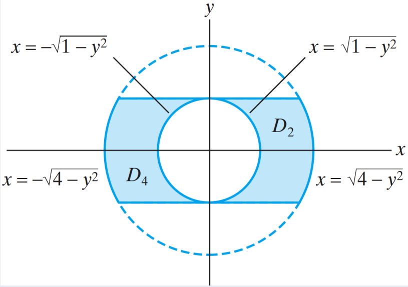


$$\int\int_{D_1} f\,dA = \int_{-\sqrt{3}}^{\sqrt{3}} \int_{1}^{\sqrt{4-x^2}} f(x,y)\,dy\,dx$$

$$\int\int_{D_3} f\,dA = \int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{4-x^2}}^{-1} f(x,y)\,dy\,dx$$

Example 7

- For the type 2 subregions

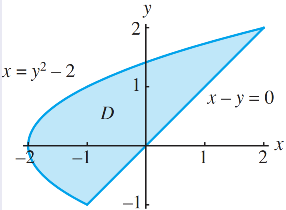


$$\int\int_{D_2} f \; dA = \int_{-1}^{1} \int_{\sqrt{1-y^2}}^{\sqrt{4-y^2}} f(x,y)dxdy$$

$$\int\int_{D_4} f \; dA = \int_{-1}^{1} \int_{-\sqrt{4-y^2}}^{-\sqrt{1-y^2}} f(x,y)dxdy$$

Example 8

- Consider the region D bounded by the line $x - y = 0$ and the parabola $x = y^2 - 2$



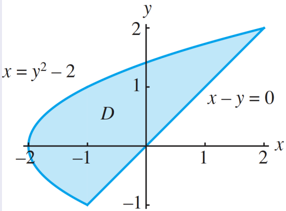
- We calculate

$$\int\int_D y \, dA$$

- In this case D is a type 2 elementary region
- The left and right boundary curves may be expressed as $x = y^2 - 2$ and $x = y$, respectively

Example 8

- Consider the region D bounded by the line $x - y = 0$ and the parabola $x = y^2 - 2$

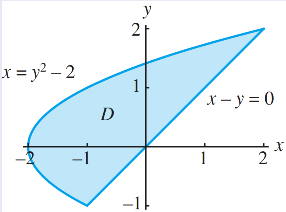


- In this case D is a type 2 elementary region
- The left and right boundary curves may be expressed as $x = y^2 - 2$ and $x = y$, respectively
- These curves intersect where

$$y^2 - 2 = y \iff y^2 - y - 2 = 0 \iff y = -1, 2$$

Example 8

- Consider the region D bounded by the line $x - y = 0$ and the parabola $x = y^2 - 2$



- Therefore, part 2. of Theorem 2.10 applies to give

$$\iint_D y \, dA = \int_{-1}^{2} \int_{y^2-2}^{y} y \, dx dy$$

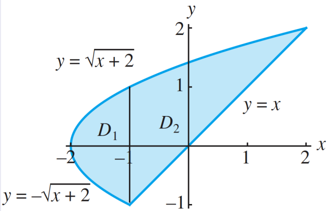
Example 8

- Therefore, part 2. of Theorem 2.10 applies to give

$$\begin{aligned}
\int\int_D y\, dA &= \int_{-1}^{2}\int_{y^2-2}^{y} y\, dxdy = \int_{-1}^{2} xy\Big|_{x=y}^{x=y^2-2} dy \\
&= \int_{-1}^{2} \left(y^2 - y^3 + 2y\right) dy = \left(\frac{y^3}{3} - \frac{y^4}{4} + y^2\right)\Big|_{y=-1}^{y=2} \\
&= \left(\frac{8}{3} - 4 + 4\right) - \left(-\frac{1}{3} - \frac{1}{4} + 1\right) = \frac{9}{4}
\end{aligned}$$

Example 8

- Note that D may be divided into two type 1 subregions along the vertical line $x = -1$



$$\begin{aligned}\int\int_D fy\ dA &= \int\int_{D_1} y\ dA + \int\int_{D_2} y\ dA \\ &= \int_{-2}^{-1}\int_{-\sqrt{x+2}}^{\sqrt{x+2}} y\ dy dx + \int_{-1}^{2}\int_{x}^{\sqrt{x+2}} y\ dy dx \\ &= \ldots = \frac{9}{4}\end{aligned}$$